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MATERIAL MOMENTUM TENSOR AND PATH-INDEPENDENT INTEGRALS OF FRACTURE MECHANICS

ALICIA GOLEBIEWSKA HERRMANN
Division of Applied Mechanics, Department of Mechanical Engineering, Stanford University, Stanford, CA 94305, U.S.A.

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Abstract—Balance laws involving the material momentum tensor are studied. By convenient choice of the Lagrangian, the balance law of moment of material momentum is derived directly by considering rotations in material space. Then the balance law connected with the property of similarity of material space is obtained. Symmetry of physical and material momenta in both representations which are used is studied.

Finally, the relations of the obtained balance laws (for the static case) to path-independent integrals of fracture mechanics are discussed.

INTRODUCTION

During the last thirty years the interest in and importance of balance (or nonconservation) laws, especially with respect to properties of material space, has increased dramatically. The large number of papers recently published on path-independent integrals in fracture mechanics (which express nothing more than nonconservation of material momentum) illustrates this trend. The study of these balance laws, their significance and range of applicability (e.g. linear vs nonlinear elasticity) appears to be of timely concern.

In a previous paper[1], the general idea of nonconservation laws for a continuum was discussed, based on symmetrical use of material and physical coordinates as dependent and independent variables. What emerged from that consideration was the importance of the material momentum tensor (on the level of the stress tensor) and the ambiguity of names: equations of motion and conservation laws, e.g. conservation of material momentum in one representation is becoming equation of motion in the other. All balance laws were obtained by simple procedures performed on the Lagrangian. However, the class of balance laws under consideration was not complete. Having in mind possible applications of path-independent integrals[2] to fracture mechanics, we would like to study other balance laws connected with rotation.

The procedure used in the paper involves the Noether theorem applied directly to dependent variables of the action. Then the comparison with broadly used path-independent integrals J and L, can be given.

The derived results apply to dynamic as well as nonlinear behavior.

ROTATIONS IN PHYSICAL AND MATERIAL SPACE; DIRECT APPROACH

It is believed that while the equations of motion can be obtained by varying the fields only, the conservation laws are derived by varying fields and independent variables: in other words we have to be concerned with a variational principle with varying boundaries. This immediately leads to the inconvenience of dealing with noncommuting operations, namely variation and differentiation of fields with respect to independent variables. However, in the case of momenta (physical or material) the distinction between field equations and conservation laws turned out to be only superficial, depending on the chosen description. This suggests the possibility of overcoming the inconvenience by proper choice of the representation used.

It is possible[3] to treat the parameters connected with transformations of interest as additional variables and corresponding conservation laws are nothing else but equations of motion related to these variables.

Let us recall here again that direct variations of dependent variables are much simpler to deal with because variations commute with differentiation. Now we can take advantage of the two descriptions used in[1], namely using the physical coordinates $\mathbf{x}$ or the material coordinates $\mathbf{X}$ as field variables, depending on desired transformations of interest. If we want to study transformations of material space, we will choose the description using X_i (and their derivatives) as dependent variables while x_i, t (t denotes the time) form a set of independent variables, so the action integral S has the form

$$S = \int \mathrm{d}t \int \mathscr{L}(x_i, t; X_k, V_k, X_{i;j})\, \mathrm{d}^3x \tag{1}$$

where

$$V_i = \frac{\partial X_i}{\partial t}, \qquad X_{i;j} = \frac{\partial X_i}{\partial x_j}.$$

By contrast, if we are interested in invariance with respect to some transformations performed on physical space, we use the representation of action

$$S = \int \mathrm{d}t \int L(X_i, t; x_k, v_k, x_{i,j})\, \mathrm{d}^3X \tag{2}$$

where

$$v_i = \frac{\mathrm{d}x_i}{\mathrm{d}t}, \qquad x_{i,j} = \frac{\partial x_i}{\partial X_j}.$$

Let us start with conservation laws related to material space, so we consider symmetries of given Lagrangian $\mathscr{L}$ used in (1):

$$\mathscr{L} = \mathscr{L}(x_i, t; X_k, V_k, X_{i;j}).$$

The infinitesimal rigid rotations in material space lead to the following transformations of X_i and their derivatives:

$$\left.\begin{aligned} X'_i &= X_i + \epsilon_{ijk}\omega_j X_k \\ X'_{i;l} &= X_{i;l} + \epsilon_{ijk}\omega_j X_{k;l} \\ V'_i &= V_i + \epsilon_{ijk}\omega_j V_k \end{aligned}\right\} \tag{3}$$

and ω_j ($j = 1, 2, 3$) are infinitesimal parameters of the transformation. The ω_j's can be treated now as additional (or virtual) variables, and the equations of motion with respect to them can be obtained directly by differentiating $\mathscr{L}$ with respect to ω_j:

$$\frac{\partial \mathscr{L}}{\partial \omega_j} = \epsilon_{ijk}\left[\frac{\partial \mathscr{L}}{\partial X_i} X_k + \frac{\partial \mathscr{L}}{\partial X_{i;p}} X_{k;p} + \frac{\partial \mathscr{L}}{\partial V_i} V_k\right]$$

(where we used the fact that for real motion $X'_i = X_i$). Using the material momentum vector B_i and the material momentum tensor B_{ik} introduced in [1] as

$$B_i = \frac{\partial \mathscr{L}}{\partial V_i}, \qquad B_{ik} = \frac{\partial \mathscr{L}}{\partial X_{i;k}} \tag{4}$$

and the equations of motion for the description (1) (see eqn (A) of [1]):

$$\frac{\partial}{\partial t} B_i + \frac{\partial}{\partial x_k} B_{ik} = \frac{\partial \mathscr{L}}{\partial X_i} \tag{5}$$

we obtain finally:

$$\frac{\partial \mathscr{L}}{\partial \omega_j} = \frac{\partial}{\partial t}(\epsilon_{ijk} B_i X_k) + \frac{\partial}{\partial x_p}(\epsilon_{ijk} B_{ip} X_k). \tag{6}$$

If $\mathscr{L}$ is invariant under rotations, $(\partial\mathscr{L}/\partial\omega_j)=0$, we derive from (6) the law of conservation of moment of material momentum

$$\frac{\partial}{\partial t}(\epsilon_{ijk}B_kX_j)+\frac{\partial}{\partial x_p}(\epsilon_{ijk}B_{kp}X_j)=0. \tag{7}$$

Physical laws do not depend on the choice of the particular reference frame or representation. Equation (7) expresses a certain conservation law in one of possible representations, namely that related to (1).

Had we started with (2), the same conservation of moment of material momentum would be mathematically related to the invariance of L under some transformations (namely rotations) performed on X_i, which now, however, belong to independent variables. The conservation laws can be derived via Noether's theorem from a variational principle with varying boundaries.

However, having derived these conservation laws in one representation, we can simply transform them into any desired other representation, without starting over with an action integral and new Lagrangian form appropriate for chosen representation.

In the case under consideration the other desirable representation is the one connected with expression (2); with X_i and t as independent variables. The passage requires two steps: first, an obvious one is connected with the change of integration variables and involves the use of the Jacobian j of that transformation:

$$j\mathscr{L}=L, \qquad j=\det(\partial x_i/\partial X_k).$$

The second is related to the change of sets of functions appearing in L and $\mathscr{L}$, e.g. it is not only that V_k appearing in $\mathscr{L}$ will become a function of X_i, t instead of x_i, t, but it has to be first expressed in terms of the new dependent variables x_k, v_k, $x_{i,j}$ (which we wish to use in L) each of them being a function of X_i, t.

Finally, we obtain from (7) the statement of the conservation of the moment of material momentum in the transformed form:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon_{ijk}b_iX_k)+\frac{\partial}{\partial X_p}(\epsilon_{ijk}b_{ip}X_k)=0 \tag{8}$$

where b_i, b_{ik} are the material momentum vector and tensor, respectively, in this representation given as:

$$b_i=\frac{\partial L}{\partial v_j}x_{j,i}, \qquad b_{ik}=\frac{\partial L}{\partial x_{j,k}}x_{j,i}-L\delta_{ik}. \tag{9}$$

If we are interested now in the invariance of S with respect to rotations of physical space, we start with expression (2). The infinitesimal rotations lead to the relations similar to (3):

$$x'_i=x_i+\epsilon_{ijk}\lambda_jx_k, \qquad x'_{i,l}=x_{i,l}+\epsilon_{ijk}\lambda_jx_{k,l}, \qquad v'_i=v_i+\epsilon_{ijk}\lambda_jv_k. \tag{10}$$

The parameters λ_k can be interpreted as field variables additional to x_i in the Lagrangian L. The corresponding Euler–Lagrange equations express conservation of moment of momentum (if $\partial L/\partial\lambda_j=0$):

$$\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon_{ijk}p_kx_j)+\frac{\partial}{\partial X_p}(\epsilon_{ijk}p_{kp}x_j)=0 \tag{11}$$

where

$$p_i=\frac{\partial L}{\partial v_i}, \qquad p_{ip}=\frac{\partial L}{\partial x_{i,p}}. \tag{12}$$

In deriving (11) we made use of equations of motion for x_i, namely (see (b) of [1]):

$$\frac{d}{dt} p_i + \frac{\partial}{\partial X_k} p_{ik} = \frac{\partial L}{\partial x_i}. \tag{13}$$

Should L be not invariant with respect to rotations in **x**-space, the r.h.s. of (11) would not be zero, but $(\partial L/\partial \lambda_i)$. Again, (11) can be transformed to the (**x**, t) space with the result:

$$\frac{\partial}{\partial t}(\epsilon_{ijk} P_k x_j) + \frac{\partial}{\partial x_p}(\epsilon_{ijk} P_{kp} x_j) = 0 \tag{14}$$

where

$$P_k = \frac{\partial \mathscr{L}}{\partial V_j} X_{j;k}, \qquad P_{ik} = \frac{\partial \mathscr{L}}{\partial X_{j;k}} X_{j;i} - \mathscr{L}\delta_{ik}. \tag{15}$$

It is of interest to establish the relations between conservation of moments of momenta and symmetries of material and physical momentum tensors. As is well-known, the conservation of moment of momentum results in the symmetry of the Cauchy stress tensor. We would like to find out whether and how relations (7), (8) and (11), (14) influence the properties of b_{ij}, B_{jk} and p_{ij}, P_{kl}.

We can easily verify that neither eqn (7) nor (11) supply any simple conditions for B_{ik} or p_{ij}, respectively. Equation (8) involving b_{ik}, for the case when L does not depend on X_i explicitly, produces a condition of interest:

$$b_{ik} = b_{ki} \tag{16}$$

because

$$\epsilon_{ijk}\left[\frac{d}{dt}(b_i X_k) + \frac{\partial}{\partial X_p}(b_{ip} X_k)\right] = \epsilon_{ijk} X_k \left[\frac{d}{dt} b_i + \frac{\partial}{\partial X_p} b_{ip}\right] + \epsilon_{ijk} b_{ik}$$

and if we use eqn (a) of [1] with $(\partial L/\partial X_i)_{\text{exp}} = 0$, the term in brackets vanishes. In the case of dependence of L on X_i, the antisymmetric part of b_{ik} is related to to the moment of material force (represented by $\partial L/\partial X_i$).

For the physical momentum, the situation is quite analogous: only one eqn (14), namely the one involving P_{ik}, supplies a simple condition, which is

$$P_{ik} = P_{ki} \tag{17}$$

provided $\mathscr{L}$ does not depend on **x** explicitly. For some special cases, however, it is possible to relate the symmetry of p_{ij} to the symmetry of P_{ik}.

From (11) we derive (taking into account (13) with $(\partial L/\partial x_i) = 0$):

$$\epsilon_{ijk}(v_j p_k + p_{kp} x_{j,p}) = 0.$$

Formulas of this type were discussed for the Eshelby tensor in [6]. For the simple case, when $p_k = \rho v_k$, the first part of the above relation vanishes. In general we can consider the static case, where $p_k = 0$. Then the above condition reduces to

$$p_{mp} x_{l,p} = p_{lp} x_{m,p}.$$

But $x_{l,p} = \delta_{lp} + u_{l,p}$, where u_l is a displacement vector. For linear elasticity with infinitesimal deformations such that all $|u_{i,j}| \ll |$, the symmetric part of p_{ip} is a linear function of $u_{i,j}$ while, as can be seen from above relation, the antisymmetric part is a quadratic function of displacement gradients, and can be neglected, so approximately:

$$p_{ik} = p_{ki}. \tag{18}$$

We have to remember, however, the restrictions under which the last relation was derived. Let us observe only that from the way p_{ip} was introduced in[1], and from the form of the equation of motion in the $\mathbf{X}, t$ representation, it is clear that p_{ip} is not an analog of the Cauchy stress tensor, but is rather related to the Piola–Kirchhoff tensor of the first type. (The relations between all known and newly introduced tensors will be the subject of a separate study.) Of course, in the static, linear elastic case the differences disappear and that is how we can easily relate the quantities introduced here to certain ones used in fracture mechanics based on linear elasticity.

Concluding this section, let us remark that relations (7), (8), (11), (14) are identical to those obtained in[4] on the basis of a general variational principle with varying boundaries. While in the former study only conservation laws were involved, the direct approach presented here generalizes them to balance laws or nonconservation laws.

MATERIAL BALANCE LAWS AND PATH-INDEPENDENT INTEGRALS OF FRACTURE MECHANICS

The balance (or conservation) laws derived in preceding sections were given in a unified differential form, putting the well-known equilibrium equations on the same level as balance laws for material momentum. They all are written in the differential, local form. There are, however, some differences in using these equations. For example, if we write

$$\sigma_{ij,j} = f_i$$

we do not mind using the local form, because f_i usually represents gravitation and we know the explicit formula for it, namely $f_i = \rho g x_3 \delta_{i3}$ where ρ is the density and we assume x_3 is directed along the line of action of the gravitation. In the case of e.g. a crack, we rather wish to know the total force on a crack or a crack tip, rather than the density of that force.

That is why we might want to use global forms of material balance laws rather than differential ones. In general, we can expect that the result of integration will be a constant if the integration is performed over all 4 independent variables, including time. By contrast, if we perform integration over space-type variables only, the possibility of obtaining path-independent quantities is rather doubtful unless, of course, we consider a static case in which there is no time-dependence. This is the situation for which the well-known path-independent integrals of fracture mechanics were established.

In order to relate our results to those integrals, we will restrict ourselves now to the static case. Then the Lagrangian L reduces to $-W$, the density of elastic strain energy, and the balance of material momentum (see (a) of [1]) takes the form

$$b_{ik;k} = j_i \tag{19}$$

where

$$b_{ik} = -\frac{\partial W}{\partial x_{j,k}} x_{j,i} + W\delta_{ik}, \quad j_i = \left(\frac{\partial W}{\partial X_i}\right)_{\exp}. \tag{20}$$

Integration of (19) over a certain volume V gives

$$\int_S b_{ik} n_k \, \mathrm{d}S = F_i \tag{21}$$

where $F_i = \int_V j_i \, \mathrm{d}V$ and S is a material surface enclosing a volume V. We can expect (21) to be independent of the surface S, as long as the defect (whose presence in material space is represented by $(\partial W/\partial X_i)$) remains inside the surface and our surface does not include other defects.

F_i strongly resembles the J integral

$$J_i = \int (Wn_i - T_j u_{j;i})\,\mathrm{d}S \tag{22}$$

where $T_j = \sigma_{kj}n_k$ are the tractions on the surface, or rather on a line because J is mostly used for a plane crack, so for a two-dimensional case. σ is a Cauchy stress tensor, defined by means of the elastic energy and strain tensor ϵ:

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}, \quad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{23}$$

To compare (21) with (22) we will express b_{ik} in terms of $u_{i,i}$ and derivatives with respect to $u_{i,j}$, rather than $x_{i,j}$, namely

$$x_{i,j} = u_{i,j} + \delta_{ik},$$

and

$$b_{ik} = W\delta_{ik} + p_{jk}u_{j,i} + p_{ik} \tag{24}$$

and p_{ik} in the static case is given in conformance with (9) as

$$p_{ik} = -\frac{\partial W}{\partial x_{i,k}} = -\frac{\partial W}{\partial u_{i,k}}.$$

To identify (21) with (26) we can see that two conditions (independently) have to be satisfied:

(1) $$p_{ik} = -\sigma_{ik}$$

(2) $$\int_S p_{ik}n_k\,\mathrm{d}S = \int_v p_{ik,k}\,\mathrm{d}V = 0.$$

The first condition requires that p_{ik} is symmetric, in other words that eqn (18) is satisfied. As it was discussed, this condition is fulfilled if the moment of momentum is conserved and if assumptions of linear elasticity are satisfied.

Condition (2), as can be seen from (13), is related to the absence of body forces. From all these restrictions it seems that (21) is a generalization of J_i as given by (22) because we can include body forces as well as their nonvanishing moments. In addition, the expression for p_{ij} does not assume linear elasticity.

We will consider now the quantity obtained by integration of eqn (8) with nonvanishing r.h.s. which we will denote by l_i

$$\int_v l_i\,\mathrm{d}V = \int_S \epsilon_{ijk}b_{kp}X_j n_p\,\mathrm{d}S. \tag{25}$$

As we mentioned before, we consider only the static case. We wish to compare (29) with L_i given by (see [2]):

$$L_i = \int \epsilon_{ijk}(WX_k n_j + T_p u_{p,j}X_k + T_j u_k)\,\mathrm{d}S. \tag{26}$$

(The relation of L_i to the virtual rotation of a crack was discussed in [11].) Under the same assumptions as for J_i, we can show the equivalence of (26) and (25), up to the sign,†

Using the form (24) we can verify by inspection that the first two terms of b_{ik} (24) in (25)

†In classical continuum mechanics balance of angular momentum is derived by equating the time rate of the angular momentum to the resultant moment of all forces: the contributions coming from tractions T_k on the surface have the form

$$\epsilon_{ijk}T_k x_j = \epsilon_{ijk}x_j\sigma_{kp}n_p.$$

Because we treat here the rotations in physical and material space, the corresponding contributions as related to the material balance of angular momentum have a form analogous to the above (compare (25)) where $\mathbf{x}$ is replaced by $\mathbf{X}$ and the stress tensor by the material momentum tensor.

become the first two terms of (26). Besides, the conservation of moment of momentum (which we have to accept in order to replace p_{ik} by $-\sigma_{ik}$) gives:

$$\int \epsilon_{ijk}p_{kp}X_jn_p \,\mathrm{d}S = -\int \epsilon_{ijk}p_{kp}u_jn_p \,\mathrm{d}S$$

because

$$\int \epsilon_{ijk}p_{kp}x_jn_p \,\mathrm{d}S = 0$$

and

$$x_j = X_j + u_j.$$

Thus the equivalence of third terms has been established.

To complete this section let us once more recall that for statics of a continuum without defects, the conservation laws connected with properties of material space can be written in global form by means of one material momentum tensor and its "product" with the material coordinate X_i as:

$$\int_S b_{ik}n_k \,\mathrm{d}S = 0, \qquad \int_S \epsilon_{ijk}X_jb_{kp}n_p \,\mathrm{d}S = 0. \tag{27}$$

They strongly resemble the conservation of linear momentum and angular momentum in physical space, respectively (if we replace X_i by x_i and b by p, respectively).

For a continuum with defects, the r.h.s. of (27) are, in general, not zero. If we consider a plane crack in an infinite homogeneously stressed medium and the integration is performed along a curve enclosing the whole crack (not a crack tip), the first relation of (27) is still satisfied, i.e. r.h.s. is still zero[11].

Concluding this section let us underline one important difference between considerations included here and in earlier sections. The first part is concerned with a general formalism and derived balance (or conservation) laws were stated in general form, assuming the action integral is based on the Lagrangians used in (1) or (2).

However, all equations are local, i.e. the formulas involve differential operators only. Passing to more specific problems of path-independent integrals used in fracture mechanics, we passed from dynamic to static consideration, and only then by integration, we obtained some balance laws in global form.

As far as path-indepencence is concerned, we do not believe that dynamic generalizations exist: we can formally integrate any time-dependent relations over arbitrary regions, however their path-independence is questionable.

The problem of similarity transformation and the related M integral will be discussed in a later paper.

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